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Multilinear Algebras and Tensors with Vector Bundles of Manifolds

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Abstract— In this paper some important aspects of tensor algebra, tensor product, exterior algebra, symmetric algebra, module of section, graded algebra, vector subbundles are studied. The purpose of this paper is to develop the theories which are based on multi-linear algebra and tensors with vector bundles of manifolds. A *Theorem* 1.34. is established by using sections and fibrewise orthogonal sections of an application of Gran-Schmidt.

Keywords: Multilinear and tensor algebra, tangent and tensor bundle, subbundles, associated frame bundles, graded and Symmetric algebra.

I. INTRODUCTION

Multilinear algebra and tensor algebra of $R-$ modules are needed to use higher order tensors. The tangent bundle, various tensor bundles, subbundles and associated frame bundles will play important roles as the theory of manifolds is developed. A theorem related with subbundle is treated with various tensors, graded algebras, tensor product, and trivial bundles.

II. TENSOR ALGEBRA

We build a universal model of multi-linear objects called the tensor algebra over R in order to study $R-$multilinear maps, , where R will be the ring $C^\infty(M)$.

Definition 1.1 [1] An $R-$module V is *free* if there is a subset $B \subset V$ such that every nonzero element $v \in V$ can be written uniquely as a finite $R-$linear combination of elements of B. The set B will be called a *(free) basis* of R.

Example 1.2 Let $\pi : E \to M$ be a trivial $n-$ plane bundle. Then $\Gamma(E)$ is a free $C^\infty(M)$ $-$module on a basis of n elements.

Example 1.3 The integer lattice $\mathbb{Z}^k$, a free $\mathbb{Z}$ $-$module is a $C^\infty(M)$ module.

Definition 1.4 If V_1, V_2, V_3 are objects in $\mathcal{M}(R)$, a map $\varphi : V_1 \times V_2 \to V_3$ is $R-linear$ if

$\varphi(.\ , V_2) : V_1 \to V_3$

$\varphi(V_1, .) : V_2 \to V_3$

are $R-$linear, $\forall\ v_i \in V_i,\ i = 1,2.$

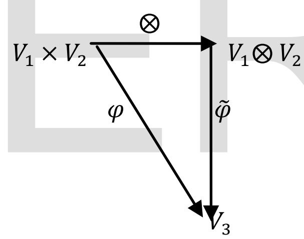
Definition 1.5 [2] A *tensor product* of $R-$modules V_1, V_2 is an $R-$module $V_1 \otimes V_2$, together with an $R-$bilinear map

$$\otimes : V_1 \times V_2 \to V_1 \otimes V_2$$

with the following "universal property":given any $R-$modules V_3 and any $R-$bilinear map

$$\varphi : V_1 \times V_2 \to V_3,$$

there is a unique $R-$linear map $\tilde{\varphi} : V_1 \otimes V_2 \to V_3$ such that the diagram



commutes. Write $\otimes(v, w) = v \otimes w$.

Corollary 1.6 If V_i is an $R-$module, $i = 1,\ 2,\ 3$, there are unique $R-$linear isomorphism

$$V_1 \otimes (V_2 \otimes V_3) = (V_1 \otimes V_2) \otimes V_3 = V_1 \otimes V_2 \otimes V_3$$

identifying

$$v_1 \otimes (v_2 \otimes v_3) = (v_1 \otimes v_2) \otimes v_3$$
$$= v_1 \otimes v_2 \otimes v_3, \quad \forall\ v_i \in V_i,\ i = 1,2,3.$$

Definition 1.7 An element $v \in V_1 \otimes ... \otimes V_k$ is said to be *decomposable* if it can be written as a monomial $v = v_1 \otimes\ ... \otimes v_k$, for suitable elements $v_i \in V_i, 1 \le i \le k$. Otherwise, v is said to be *indecomposable*.

Lemma 1.8 If V and W are $R-$modules with respective bases A and B, then $V \otimes W$ is free with basis $C = \{a \otimes b \mid a \in A, b \in B\}$.

Proof. An arbitrary element $v \in A \otimes B$ can be written as a linear combination of decom-posable. A decomposable element $V \otimes W$ can be expanded the multilinearity of tensor product, to a linear

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combination of elements of C, proving that C spans $V \otimes W$. It remains to show that, if

$$\sum_{i,j=1}^{p,q} c_{ij}\, a_i \otimes b_j = \sum_{i,j=1}^{p,q} d_{ij}\, a_i \otimes b_j,$$

where $a_i \in A$ and $b_j \in B$, $1 \leq i \leq p$, $1 \leq j \leq q$ then all $c_{i,j} = d_{i,j}$. Subtracting one expression from the other, we only need to prove that

$$\sum_{i,j=1}^{p,q} c_{ij}\, a_i \otimes b_j = 0$$

implies that all $c_{i,j} = 0$. The bilinear functional $\varphi : V \times W \to R$ corresponds one to one to any functions $f : A \times B \to R$.The correspondence is $\varphi \leftrightarrow \varphi \mid (A \times B)$. Thus, the linear functional $\tilde{\varphi} : V \otimes W \to R$ also corresponds one to one to these functions $f : A \times B \to R$.

If $(a, b) \in (A \times B)$, let $f_{a,b} : (A \times B) \to R$ be the function taking the value 1on (a,b) and the value 0 on every other element of $(A \times B)$.The corresponding linear functional will be denoted by $\tilde{\varphi}_{a,b}$. Applying $\tilde{\varphi}_{a_i,b_j}$ to equation (1.1), we see that all $c_{ij} = 0$. This completes the proof.

Proposition 1.9 If $\lambda_i : V_i \to W_i$ is an $R-$linear map, $1 \leq i \leq k$, there is a unique $R-$linear map

$$\lambda_1 \otimes \ldots \ldots \otimes \lambda_k : V_1 \otimes \ldots \ldots \otimes V_k \to W_1 \otimes \ldots \ldots \otimes W_k$$

which, on decomposable elements, has the formula

$$(\lambda_1 \otimes \ldots \ldots \otimes \lambda_k)\,(v_1 \otimes \ldots \ldots \otimes v_k) = \lambda_1(v_1) \otimes \ldots \ldots \otimes \lambda_k(v_k).$$

Proof. We know the decomposable span. So, the uniqueness is immediate. For existence, let us define the multilinear map

$$\lambda : V_1 \times \ldots \ldots \times V_k \to W_1 \otimes \ldots \ldots \otimes W_k$$

by

$$\lambda(v_1, \ldots \ldots, v_k) = \lambda_1(v_1) \otimes \ldots \ldots \otimes \lambda_k(v_k).$$

Then $\lambda_1 \otimes \ldots \ldots \otimes \lambda_k$ is defined to be the unique associated linear map. Hence, the proof is complete.

Definition 1.10 For the module of $R-$linear functionals, the *dual* V^* of an $R-$module V is $Hom_R(V, R)$.

Lemma 1.11 If V has a finite free basis $\{v_1, \ldots \ldots \ldots, v_n\}$, then V^* has a finite free basis $\{v_1, \ldots \ldots \ldots, v_n\}$, called the basis and defined by

$$v_i^*(v_j) = \delta_j^i, \quad 1 \leq i, j \leq n.$$

Corollary 1.12 If $V_1, \ldots \ldots, V_k$ are free $R-$modules on bases $B_1, \ldots \ldots, B_k$, respectively, then $V_1 \otimes \ldots \ldots \otimes V_k$ is a free $R-$module with basis

$$B = \{v_1 \otimes \ldots \ldots \otimes v_k \mid v_i \in B_i,\ 1 \leq i \leq k\}.$$

Proposition 1.13 There is a unique $R-$linear map

$$l : V_1^* \otimes \ldots \ldots \ldots \otimes V_k^* \to (V_1 \otimes \ldots \ldots \otimes V_k)^*$$

which on decomposable elements has the formula

$$l(\eta_1 \otimes \ldots \ldots \otimes \eta_k)\,(v_1 \otimes \ldots \ldots \otimes v_k) = \eta_1(v_1) \otimes \ldots \ldots \otimes \eta_k(v_k).$$

If the $R-$modules V_i are all free on finite bases, then l is a canonical isomorphism.

Proof. Since the decomposable span, uniqueness is immediate. For existence, consider the multi linear functional

$$\theta : V_1^* \times \ldots \ldots \ldots \times V_k^* \times V_1 \times \ldots \ldots \times V_k \to R$$

by

$$\theta(\eta_1, \ldots \ldots, \eta_k, v_1 \ldots \ldots, v_k) = \eta_1(v_1) \ldots \ldots \eta_k(v_k).$$

by the universal property, this gives the associated linear functional

$$\tilde{\theta} : V_1^* \otimes \ldots \ldots \ldots \otimes V_k^* \otimes V_1 \otimes \ldots \ldots \otimes V_k \to R,$$

and we define

$$l : V_1^* \otimes \ldots \ldots \ldots \otimes V_k^* \to (V_1 \otimes \ldots \ldots \otimes V_k)^*$$

by

$$l(\eta)(v) = \tilde{\theta}\,(\eta \times v).$$

If $\{v_{i,1}, \ldots \ldots, v_{i,m_i}\}$ is a basis of $V_i, 1 \leq i \leq k$, let $\{v_{i,1}^*, \ldots \ldots, v_{i,m_i}^*\}$ be the dual basis. Let B and B^*be the respective bases of $V_1 \otimes \ldots \ldots \otimes V_k$ and $V_1^* \otimes \ldots \ldots \ldots \otimes V_k^*$ given by the Corollary 1.11. The formula

$$l(v_{1,j_1}^* \otimes \ldots \ldots \otimes v_{k,jk}^*)(v_{1,i_1} \otimes \ldots \ldots \otimes v_{k,i_k}) = \delta_{i_1}^{j_1} \ldots \ldots \delta_{i_k}^{j_k} = \delta_{i_1 \ldots \ldots \ldots i_k}^{j_1 \ldots \ldots j_k}$$

shows that l carries the basis B^* one to one onto the basis dual to B, so l is an isomorphism. This completes the proof.

Definition 1.14 [3] A *graded (associated) algebra* A over R is a sequence $\{A^n\}_{n=0}^{\infty}$ of $R-$modules, together with $R-$bilinear maps (multiplication)

$$A^n \times A^m \to A^{n+m}, \quad \forall\, n, m \geq 0,$$

which is strongly associative in the sense that the compositions

$$(A^n \times A^m) \times A^r \xrightarrow{.\times id} A^{n+m} \times A^r \xrightarrow{\cdot} A^{n+m+r}$$

$$A^n \times (A^m \times A^r) \xrightarrow{id \times .} A^n \times A^{m+r} \xrightarrow{\cdot} A^{n+m+r}$$

are equal, $\forall\, n, m, r \geq 0$.

Definition 1.14 The graded algebra A is *connected* if $A^0 = R$ and

$$A^0 \times A^m \xrightarrow{\cdot} A^m \xleftarrow{\cdot} A^m \times A^0$$

are equal to scalar multiplication, $\forall\, m \geq 0$.

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Definition 1.15 If V is an R −module, then $\mathcal{T}(V)$ with multiplication $\otimes$, is called the *tensor algebra* of V. It is clear that the tensor algebra $\mathcal{T}(V)$ is connected.

Definition 1.16 A *homomorphism* $\varphi : A \to B$ of graded R- algebras is a collection of R- linear maps $\varphi^n : A^n \to B^n, \forall\, n \geq 0$, such that the diagrams

$$\begin{array}{ccc} A^n \times A^m & \longrightarrow & A^{n+m} \\ \varphi^n \times \varphi^m \downarrow & & \downarrow \varphi^{n+m} \\ B^n \times B^m & \longrightarrow & B^{n+m} \end{array}$$

commute, $\forall\, n,\ m \geq\ 0$. The homomorphism φ is an isomorphism if φ^n is bijective, $\forall\, n \geq 0$.

Theorem 1.17 If $\lambda : V \to W$ is an R −linear map, then there is a unique induced homomorphism $\mathcal{T}(\lambda){:}\mathcal{T}(V) \to \mathcal{T}(W)$ of graded R −algebras such that $\mathcal{T}^0(\lambda) = id_R$ and $\mathcal{T}^1(\lambda) = \lambda$. This homo-orphism satisfies

$$\mathcal{T}^n(\lambda)(v_1 \otimes v_2 \otimes \ldots\ldots \otimes v_n) = \lambda\,(v_1) \otimes \lambda\,(v_2) \otimes\ \ldots\ldots \otimes \lambda(v_n),$$

$$\forall\, n \geq 2,\ \forall\, v_i \in V,\ 1 \leq i \leq n.$$

Finally, this induced homomorphism makes $\mathcal{T}$ a covariant function from the category of R −modules R −linear maps to the category of graded algebras over R and graded algebra homomorphisms.

Definition 1.18 The space of tensors on V of type (r, s) is the *tensor product*

$$\mathcal{T}_s^{\,r}(V) = \mathcal{T}_0^{\,r}(V) \otimes\ \mathcal{T}_s^{\,0}(V).$$

A tensor $\alpha\,\epsilon\,\mathcal{T}_s^{\,r}(V)$ is said to have *covariant degree* r and contravariant degree s.

III. EXTERIOR ALGEBRA

Let R be any commutative ring with unity 1 such that $\frac{1}{2}\,\epsilon\, R$. That is, if $2 = 1 + 1\,\epsilon\, R$, then $\frac{1}{2}\,\epsilon\, R$ has the property that $\frac{1}{2}\cdot 2 = 1$. In the case that $R = \mathbb{F}$ is a field, this means that the characteristic of $\mathbb{F}$ is not 2.

Definition 1.19[4] *The exterior algebra* of V is the connected graded R-algebra

$$\Lambda(\mathrm{V}) = \{\Lambda^k(V)\}_{\mathrm{k}=0}^{\infty}$$

with multiplication
$\Lambda^p(V) \times \Lambda^q(V) \overset{\Lambda}{\to} \Lambda^{p+q}(V)$

where, the R −module $\Lambda^k(V)$ is the k th exterior power of V.

Lemma 1.120 Let V be an R −module, $v \in V$. Then $v = -v \Leftrightarrow v = 0$.

Proof. Let V be an R −module where $v \in V$. Then

$v = 0 \Rightarrow v = -v.$

For the converse

$v = -v \Rightarrow 2v = 0$

$\Rightarrow v = 1/2(2v)$

$\Rightarrow v = 1/2(0)$

$\therefore v = 0.$

This completes the proof.

Definition 1.21 Let V and W be R- modules. An antisymmetric *$K -$ linear map* $\varphi :\ V^k \to W$ is a $K -$ linear map such that

$$\varphi\big(v_{\sigma(1)}, \ldots\ldots\ldots, v_{\sigma(k)}\big) = (-1)^{\sigma}\ \varphi\ (v_1, v_2, \ldots\ldots, v_k),$$

$$\forall\ v_1, v_2, \ldots\ldots, v_k \in V, \forall\ \sigma \in \textstyle\sum k$$

where $(-1)^{\sigma} = \begin{cases} 1, & \sigma \text{ an even permutation,} \\ -1, & \sigma \text{ an odd permutation.} \end{cases}$

Lemma 1.22 If $\varphi :\ V^k \to W$ is antisymmetric, then $\widetilde{\varphi}\left(\mathfrak{A}^k(V)\right) = \{0\}$.

Proof. It will be enough to show that $\widetilde{\varphi}$ vanishes on a set spanning $\mathfrak{A}^k(V)$. Thus, if $w \in \mathcal{T}^p(V)\ u \in \mathcal{T}^q(V),\ p + q = k - 2,$ and $v_1, v_2 \in V$, we will show that

$$\widetilde{\varphi}\ (w \otimes (v_1 \otimes v_2 + v_2 \otimes v_1) \otimes u) = 0.$$

But the antisymmetry of φ implies that

$$\widetilde{\varphi}\ (w \otimes v_1 \otimes v_2 \otimes\ u) = -\widetilde{\varphi}\ (w \otimes v_2 \otimes v_1 \otimes u),$$

and the assertion follows the linearity.

Definition 1.23 An element $w \in \Lambda^k(V)$ that can be expressed in the form $v_1 \wedge v_2 \wedge \ldots \ldots \wedge v_k$, where $v_i \in V,\ 1 \leq i \leq k,$ is said to be *decomposable*. Otherwise, w is *indecomposable*.

Definition 1.24 A graded algebra A is *anticommutative* if $\alpha \in A^k$ and $\beta \in A^r \Rightarrow \alpha\beta = (-1)^{kr}\beta\alpha.$

Corollary 1.25 [5] The graded algebra $\Lambda(\mathrm{V})$ is anticommutative.

Proof. It is enough to verify the Definition 1.20 for decomposable elements of $\Lambda^k(V)$and $\Lambda^r(V)$. But that case is an elementary consequence of the case $k = r = 1$, and this latter case is given by

$$\begin{aligned} v \wedge w &= v \otimes w + \mathfrak{A}^2(V) \\ &= w \otimes v + \mathfrak{A}^2(V) \\ &= -w \wedge v, \end{aligned}$$

$\forall\ v, w \in V$. Thus the graded algebra $\Lambda(\mathrm{V})$ is anticommutative.

Corollary 1.26 If $w \in \Lambda^{2r+1}(V)$, then $w \wedge w = 0$.

Proof. Let $w \in \Lambda^{2r+1}(V)$. Then

$$w \wedge w = (-1)^{(2r+1)(2r+1)}(w \wedge w)$$

$$= w \wedge w$$

Now, by using Lemma 1.17., we have

$w \wedge w = 0$.

This completes the proof

Lemma 1.27 If $\lambda : V \to V$ is linear, then $\Lambda^m(\lambda) : \Lambda^m(V) \to \Lambda^m(V)$ is multiplication by $\det(\lambda)$.

Proof. Relative to a basis $\{e_1, \dots\dots, e_m\}$ of V, write

$$\lambda(e_i) = \sum_{j=1}^{m} a_i^j e_j, \quad 1 \le i \le m$$

then,

$$\Lambda^m(\lambda)(e_1 \wedge \dots\dots \wedge e_m) = \lambda(e_1) \wedge \dots\dots \wedge \lambda(e_m)$$

$$= \left(\textstyle\sum_{j=1}^m a_1^j e_j\right) \wedge \dots\dots \wedge \left(\textstyle\sum_{j=1}^m a_m^j e_j\right)$$

$$= \textstyle\sum_{1 \le j_1, \dots\dots, j_m \le m} a_1^{j_1} \dots\dots a_m^{j_m}\, e_{j_1} \wedge \dots\dots \wedge e_{j_m}.$$

Any term with a repeated j *index* vanishes. If $J = (j_1, j_2, \dots\dots, j_m)$ contains no repetitions, there is a unique permutation $\sigma j \in \sum m$ such that

$$j_{\sigma j}(r) = r, \; 1 \le r \le m.$$

Thus,

$$\Lambda^m(\lambda)(e_1 \wedge \dots\dots \wedge e_m)$$

$$= \left(\textstyle\sum_{\sigma \in \sum m} (-1)^\sigma a_{\sigma(1)}^1 \dots\dots a_{\sigma(m)}^m\right) e_1 \wedge \dots\dots \wedge e_m$$

$$= \det(\lambda)(e_1 \wedge \dots\dots \wedge e_m).$$

Hence, the proof is complete.

Lemma 1.28 If R is a field, a set of vectors $w_1, w_2, \dots\dots, w_k \in V, k \ge 2$, is linearly independent if and only if $w_1 \wedge w_2 \wedge \dots\dots \wedge w_k \ne 0$.

Proof. If R is a field then consider the set of vectors $w_1, w_2, \dots\dots, w_k \in V$, $k \ge 2$. Again if the set is dependent, the existence of universe in R allows us to assume, without loss of generality, that

$$w_1 = \sum_{i=2}^{k} a_i w_i.$$

Then

$$w_1 \wedge w_2 \wedge \dots\dots \wedge w_k = \textstyle\sum_{i=2}^k a_i w_i \wedge w_2 \wedge \dots\dots \wedge w_k = 0.$$

Conversely, if the set is linearly independent, extend it to a basis by suitable choices of $w_{k+1}, \dots\dots, w_m \in V$. Then, we have

$$w_1 \wedge w_2 \wedge \dots\dots \wedge w_k \wedge \dots\dots \wedge w_m$$

is a basis of the one-dimensional space $\Lambda^m(V)$, hence is not 0.

This completes the proof.

Lemma 1.29 If V is a free R −module on a finite basis, then each A^k is one to one, hence $A : \Lambda(V) \hookrightarrow \mathcal{T}(V)$ is a canonical graded linear imbedding.

Proof. Let $\{e_1, \dots\dots, e_m\} \subset V$ be a basis and consider the basis

$$\{e_{i_1} \wedge \dots\dots \wedge e_{i_k}\}_{1 \le i_1 \le \cdots\dots < i_k \le i_m}$$

of $\Lambda^k(V)$. Let $\{e_1^*, \dots\dots, e_k^*\} \subset V^*$ be the dual basis. Since $\mathcal{T}^k(V^*) = \mathcal{T}^k(V)^*$, we obtain a subset

$$\{e_{j_1}^* \otimes \dots\dots \otimes e_{j_k}^*\}_{1 \le j_1 < \cdots\dots < j_k \le j_m} \subset \mathcal{T}^k(V)^*,$$

which is a part of a free basis. Then, since $j_1 < \cdots < j_k$ and $i_1 < \cdots < i_k$,

$$(e_{j_1}^* \otimes \dots\dots \otimes e_{j_k}^*)(A^k(e_{i_1} \wedge \dots\dots \wedge e_{i_k}))$$

$$= (e_{j_1}^* \otimes \dots\dots \otimes e_{j_k}^*)\left(\textstyle\sum_{\sigma \in \sum k} (-1)^\sigma e_{i_{\sigma(1)}} \otimes \dots\dots \otimes e_{i_{\sigma(k)}}\right)$$

$$= (e_{j_1}^* \otimes \dots\dots \otimes e_{j_k}^*)(e_{i_1} \otimes \dots\dots \otimes e_{i_k})$$

$$= \delta_{i_1 \dots i_k}^{j_1 \dots j_k}$$

and the assertion follows.

IV. SYMMETRIC ALGEBRA

A K −linear map $\varphi : V^k \to W$ is symmetric if, for each $\sigma \in \sum k$,

$$\varphi\left(v_{\sigma(1)}, \dots\dots, v_{\sigma(k)}\right) = \varphi(v_1, v_2, \dots\dots, v_k), \forall\, v_1, v_2, \dots\dots, v_k \in V.$$

In the usual way, we build a universal, symmetric, K −linear map

$$V^k \xrightarrow{\cdot} \mathfrak{A}^k(V),$$

usually written with the dots

$$(v_1, v_2, \dots\dots, v_k) \longmapsto v_1 v_2 \dots\dots v_k.$$

Definition 1.30 [6] The space $\mathfrak{A}^k(V)$ is called the *k th symmetric power* of V, where, as usual, $\mathfrak{A}^0(V) = R$ and $\mathfrak{A}^1(V) = V$. The connected, graded algebra $\mathfrak{A}(V) = \{\mathfrak{A}^k(V)\}_{k=0}^{\infty}$, with multiplication ".", is called the *symmetric algebra* of V.

Definition 1.31 Let V be a finite dimensional vector space over a field $\mathbb{F}$. A function $f : V \to \mathbb{F}$ is a *homogeneous polynomial* of degree k on V if, related to some basis $\{e_1, \dots\dots, e_m\}$ of V,

$$f\left(\sum_{i=1}^{m} x_i e_i\right) = P(x_1, \dots\dots, x_m)$$

is a homogeneous polynomial of degree k in the variables $x_1, \dots\dots, x_m$. The vector space of all homogeneous polynomials of degree k on V will be denoted by $P^k(V)$.

V. THE MODULE OF SECTIONS

We are going to view the set of all vector bundles over a fixed manifold M[5] as the objects of a category V_M. Let
$\pi : E \to M$

$\rho : F \to M$

be vector bundles differing fibers dimensions. A homomorphism of the $n-$plane bundle E to the $m-$plane bundle F is denoted by HOM (E, F) is naturally called $C^\infty(M) -$ module.

Theorem 1.32[7] The $C^\infty(M)-$linear map α is a canonical isomorphism of $C^\infty(M) -$ modules.

$\Gamma(E) \otimes_{C^\infty(M)} \Gamma(F) = \Gamma(E \otimes F)$.

Corollary 1.30[7] There are canonical iso- morphisms $C^\infty(M) -$ modules

$\Gamma(\mathcal{T}^k(E)) = \mathcal{T}^k(\Gamma(E))$

$\Gamma(\Lambda^k(E)) = \Lambda^k(\Gamma(E))$

$\Gamma(S^k(E)) = S^k(\Gamma(E))$.

Proof. The first part of these identities is an immediate consequence of theorem 1.29. There is canonical inclusion

$A^k : \Lambda^k(\Gamma(E)) \hookrightarrow \mathcal{T}^k(\Gamma(E))$

$A^k : \Gamma(\Lambda^k(E)) \hookrightarrow \Gamma(\mathcal{T}^k(E))$.

The second part comes from the bundle inclusions. The images of these inclusions correspond perfectly under the identification $\mathcal{T}^k(\Gamma(E)) = \Gamma(\mathcal{T}^k(E))$, proving the second identity. Similarly the third part can be proof which is same as proof of second part.

Lemma 1.33 If F and E are trivial bundles, then α is an isomorphism of $C^\infty(M) -$ modules.

Proof. In this case we choose the global sections $\{\sigma_1, \dots\dots, \sigma_n\}$ of E and $\{\mathcal{T}_1, \dots\dots, \mathcal{T}_m\}$ of F which trivialize these bundles. These are free bases of the respective $C^\infty(M) -$ modules $\Gamma(E)$ and $\Gamma(F)$, so

$\{\sigma_i \otimes_{C^\infty(M)} \mathcal{T}_j\}_{i,j=1}^{n,m}$

is a free basis of $\Gamma(E) \otimes_{C^\infty(M)} \Gamma(F)$. The set

$\{\sigma_i \otimes \mathcal{T}_j\}_{i,j=1}^{n,m}$

of point wise tensor products of sections trivializes the bundle $E \otimes F$, hence this is also a free basis of $\Gamma(E \otimes F)$. Since

$\alpha\left(\sigma_i \otimes_{C^\infty(M)} \mathcal{T}_j\right) = \sigma_i \otimes \mathcal{T}_j,$

for all relevant indices, we see that α is an isomorphism of $C^\infty(M) -$ modules. This completes the proof.

Theorem 1.34 If $F \subseteq E$ is a vector subbundle and if there is given Riemannian metric on E, then the subset $\tilde{F} \subseteq E$, fiber wise perpendicular to F, is a subbundle.

Proof. Here the local triviality all that needs to be proven. There are sections $\sigma_1, \dots\dots, \sigma_r, \sigma_{r+1}, \dots\dots, \sigma_n$ of $E|U$, trivializing that bundle, where U is a neighborhood of an arbitrary point of M. These can be chosen so that $\sigma_1, \dots\dots, \sigma_r$ are sections of $F|U$ which trivialize that bundle an application of Gran-Schmidt turns these into fiberwise orthonormal sections $S_1, \dots\dots, S_r, S_{r+1}, \dots\dots, S_n$ with the same properties. It follows that $S_{r+1}, \dots\dots, S_n$ are trivializing sections of $\tilde{F}|U$, proving that $\tilde{F}$ is a subbundle of E. Hence the proof is complete.

VI. CONCLUSION

A theorem 1.34 is established which is related with a Riemannian metric on the bundle $M \times V$. For each $x \in M$, let $\tilde{E}_x \subset \{x\} \times V$ be the subspace orthogonal to $E_x^\perp$. Consequently the set $\tilde{E} = \bigcup_{x \in M} \tilde{E}_x$ is a subbundle of $M \times V$. Also this theorem will follow form a theorem in dimension theory.

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